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THE STRUCTURE OF ROLL WAVES IN TWO-LAYER FLOWS†

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The structure of non-linear waves in a two-layer flow of an incompressible fluid in extended channels is investigated. Periodic discontinuous solutions, describing roll waves of finite amplitude, are constructed for the equations of two-layer shallow water. "Anomalous" waves of limited amplitude are found which correspond to the transition from stratified to slug flow conditions.

Like single-layer flow in an inclined channel [1], the uniform flow of a two-layer fluid can become unstable. As a result of the development of instability at the interface of the fluids, quasiregular waves of finite amplitude develop which have become known as roll waves. In the shallow-water approximation, roll waves are a subclass of periodic, discontinuous, travelling waves in which a smooth transition from subcritical flow to supercritical flow occurs in a system of coordinates moving at the velocity of the wave. The conditions for the development and the structure of roll waves in gas–liquid flows have been investigated in [2–6]. In the case of the equations of single-layer shallow water in an open inclined channel, a criterion for the non-linear stability of a wave packet of finite amplitude has been proposed in [7], based on the property of the hyperbolic form of the equations of the modulations for a two-parameter family of roll waves.

In this paper we investigate the structure of travelling waves in two-layer flow and, in particular, we describe new "anomalous" flow conditions, the transition to which substantially changes the flow pattern. It is shown that, as in the case of flows in open channels, the finite-amplitude roll waves which arise in the two-layer fluid flow in a horizontal channel form a two-parameter family. The equations of the modulations in the case of packets of roll waves can therefore be obtained by a method which is similar to that used previously [7, 8] when analysing the non-linear stability of periodic waves in open inclined channels. In this case, the condition for the stability of the wave packet are expressed in terms of the hyperbolic form of the corresponding systems of modulation equations.

1. THE EQUATIONS OF TWO-LAYER SHALLOW WATER WITH FRICTION

In long channels, friction at the walls of the channel and at the interface of the fluids has a main effect on the structure of the waves in a two-layer fluid. Using the hypothesis that the turbulent friction coefficient is constant, we write the equations of two-layer flow in a channel of constant depth in the Boussinesq approximation $0 < (\rho^- - \rho^+)/\rho_0 \ll 1$) in the form

$$h_t + (hu^-)_x = 0 \quad (H-h)_t + ((H-h)u^+)_x = 0$$
$$u_t^- + u^- u_x^- + bh_x + \rho_0^{-1} p_x^+ = -c_w \frac{u^- |u^-|}{h} - c_i \frac{(u^- - u^+)|u^- - u^+|}{h} \tag{1.1}$$
$$u_t^+ + u^+ u_x^+ + \rho_0^{-1} p_x^+ = -c_w \frac{u^+ |u^+|}{(H-h)} + c_i \frac{(u^- - u^+)|u^- - u^+|}{(H-h)}$$

Here u^+, u^- is the velocity, ρ^+ and ρ^- are the densities of the upper and lower layers, respectively, ρ_0 is the mean density, $b = (\rho^- - \rho^+)g/\rho_0$; h is the depth of the lower layer, H is the total depth of the channel, and c_w and c_i are the constant coefficients of friction on the walls of the channel and at the

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interface of the fluids. After changing to dimensionless variables, it can be assumed that $H = 1, b = 1, c_w = 1, c = c_i/c_w$ and, for the required variables h, $\gamma = u^- - u^+$, we obtain the inhomogeneous system of equations

$$h_t + (h(1-h)\gamma)_x = 0$$

$$\gamma_t + (u^{-2}/2 - u^{+2}/2 + h)_x = F \qquad (1.2)$$

where

$$F = \frac{u^+|u^+|}{1-h} - \frac{u^-|u^-|}{h} + c\frac{(u^+ - u^-)|u^+ - u^-|}{h(1-h)} \qquad (1.3)$$

The values of $u^\pm$ are determined as functions of h and γ from the following relations

$$hu^- + (1-h)u^+ = u_m \equiv \text{const}$$

$$u^+ = u_m - \gamma h, \quad u^- = u_m + (1-h)\gamma \qquad (1.4)$$

The behaviour of the characteristics and properties of the discontinuous solutions for system (1.2) have been investigated in [9, 10]. We shall therefore use these results in the analysis of the discontinuous waves.

2. TRAVELLING WAVES

We will confine ourselves to considering waves which travel to the right, that is, we consider the solutions of system (1.1) which depend on the variable $\zeta = x - Dt, D > 0$, such that $0 < u^- < D, 0 < u^+ < D$. The equations of the waves can be represented in the form

$$h(D - u^-) = m^- \; (1-h)(D - u^+) = m^+$$

$$J = (D - u^-)^2/2 - (D - u^+)^2/2 + h, \quad dJ/d\xi = F \qquad (2.1)$$

By virtue of relations (1.3), (1.4) and (2.1), the quantities J and F are functions of the single variable h.

System (2.1) reduces to the ordinary differential equation

$$\frac{dh}{d\xi} = \frac{F(h)}{\Delta(h)}, \quad \Delta(h) = 1 - \frac{m^{-2}}{h^3} - \frac{m^{+2}}{(1-h)^3} \qquad (2.2)$$

A single extremum point $h = h_i$ of the function $\Delta(h)$, corresponding to the maximum of this function, exists in the interval (0, 1). Periodic solutions, describing roll waves, can be constructed if the following condition is satisfied

$$\Delta(h_i) > 0 \qquad (2.3)$$

In this case, the graph of the function $J(h)$ is shown in Fig. 1(a). The function $J(h)$ has a local minimum at the point y and a local maximum at the point y_*. In a system of coordinates which moves at the velocity of the wave D, subcritical flow ($\Delta(h) > 0$) occurs when $y < h < y_*$ and supercritical flow ($\Delta(h) < 0$) occurs when $0 < h < y$ and when $y_* < h < 1$.

Note that it is sufficient to consider roll waves in the neighbourhood of the point y which correspond to the transition ABE. The similar wave pattern A′B′E in the neighbourhood of the point y_* can be obtained by mirror reflection of the wave profile considered below with respect to the midplane of the channel.

The construction of roll waves in the case of two-layer flow is similar to the construction carried out previously for the flow of a single-layer fluid in an inclined channel. The conditions in the discontinuity in the case of (1.1) take the form $J(h^+) = J(h^-)$ (the transition AB in Fig. 1a). Since it follows from the conditions for the stability of a discontinuity that the state ahead of the discontinuity corresponds to supercritical flow and the state behind the discontinuity corresponds to subcritical flow, then $h^+ < y < h^- < y_*$, and the necessary condition for a continuous solution, connecting these states, to exist is that the right-hand side of Eq. (2.2) should vanish at the point y, i. e. $F(y) = 0$.

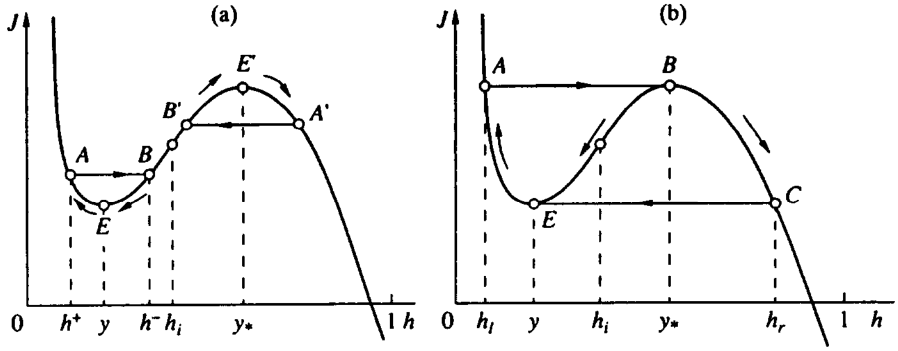

Fig. 1

Since $\Delta(h) < 0$ when $h^+ < h < y$ and $\Delta(h) > 0$ when $y < h < h^-$, then, to construct a monotonically increasing continuous solution of Eq. (2.2), connecting the two levels $h = h^+$ and $h = h^-$, it is necessary that

$$F(h) < 0 \quad \text{when} \quad h^+ < h < y \quad \text{and} \quad F(h) > 0 \quad \text{when} \quad y < h < h^- \tag{2.4}$$

Satisfaction of the conditions

$$F(y) = 0, F'(y) > 0 \tag{2.5}$$

is therefore the sufficient condition for small-amplitude roll waves to exist in the neighbourhood of a uniform flow of depth y.

When constructing finite-amplitude waves it is necessary to analyse the behaviour of the function $F(h)$ and satisfy conditions (2.4) as a function of the flow parameters. We shall show that, as in the case of single-layer flow, roll waves form a two-parameter family of solutions in the case of a fixed overall flow rate through a cross-section of the channel.

3. ROLL WAVE PARAMETERS

Suppose the quantity u_m is specified. We shall determine the flow parameters corresponding to the critical depth y. We fix $y \in (0, 1)$ and denote the velocities of the lower and upper layers in the critical cross-section by u_c, w_c. Then, the first equation in (1.4) and the conditions for the flow to be critical $\Delta(y) = 0, F(y) = 0$ in the variables

$$v = u_c/u_m, \; z = w_c/u_m, \; \theta = D/u_m$$

take the form

$$\begin{gathered} yv + (1-y)z = 1 \\ \frac{(\theta - v)^2}{y} + \frac{(\theta - z)^2}{1-y} = \frac{1}{u_m^2} \\ \frac{z^2}{1-y} - \frac{v^2}{y} + c\frac{(z-v)|z-v|}{y(1-y)} = 0 \end{gathered} \tag{3.1}$$

The last equation in (3.1) gives the equation

$$\frac{a^2}{1-y} - \frac{1}{y} + c\frac{(a-1)|a-1|}{y(1-y)} = 0 \tag{3.2}$$

in the unknown $a = z/v$.

For a given $y \in (0, 1)$, it is required to find a positive solution of this equation. Suppose $a \geqslant 1$. Then, Eq. (3.2) reduces to a quadratic equation, a unique solution of which $a \geqslant 1$ exists if and only if $0 < y \leqslant 1/2$. It can be shown in a similar manner that a unique solution of Eq. (3.2) exists in the interval (0, 1) if and only if $1/2 < y < 1$. Hence, a unique positive solution of Eq. (3.2) exists for any $y \in (0, 1)$.

Suppose $0 < y \leqslant 1/2$. Then, $a \geqslant 1$ is determined from (3.2), that is, $z \geqslant \upsilon$ or $w_c \geqslant u_c$. In this case

$$\Delta'(y) = 3\left(\frac{(D-u_c)^2}{y^2} - \frac{(D-w_c)^2}{(1-y)^2}\right) \geqslant 12((D-u_c)^2 - (D-w_c)^2) > 0$$

Similarly, when $y = y_* > 1/2$, we have $w_c < u_c$ and

$$\Delta'(y_*) = 3\left(\frac{(D-u_c)^2}{y_*^2} - \frac{(D-w_c)^2}{(1-y_*)^2}\right) \leqslant 12((D-u_c)^2 - (D-w_c)^2) < 0$$

Hence, we have shown that a local minimum of the function $J(h)$ is reached at the point $h = y$ when $y < 1/2$. When $y = y_* > 1/2$, a local maximum is reached at the critical point (Fig. 1a). Next, from system (3.1) using the known values of y and a (or y_* and a), υ and z and, consequently, u_c and w_c are found uniquely from the formulae

$$\upsilon = \frac{1}{y + (1-y)a}, \quad z = \upsilon a$$

To determine the value of θ, it is necessary to find the roots of the quadratic polynomial

$$P(\theta) = \frac{(\theta - \upsilon)^2}{y} + \frac{(\theta - z)^2}{1-y} - \frac{1}{u_m^2}$$

The minimum value of $P(\theta)$ is reached at the point $\theta = \theta_m = \upsilon(1-y) + zy$. Therefore, when $y < 1/2$, it follows from the inequality $\upsilon < z$ that $\upsilon < \theta_m < z$ and, when $y_* > 1/2$, it follows from the inequality $z < \upsilon$ that $z < \theta_m < \upsilon$. Since a solution of the equation $P(\theta) = 0$ is sought such that $z < \theta$, $\upsilon < \theta$, then $\theta_m < \theta$ and the solution is defined uniquely. When $y < 1/2$, a solution only exists when $P(z) < 0$, that is, when

$$u_m < \frac{\sqrt{y}}{z - \upsilon} \tag{3.3}$$

When $y_* > 1/2$, satisfaction of the inequality $P(\upsilon) < 0$, that is,

$$u_m < \frac{\sqrt{1-y_*}}{\upsilon - z} \tag{3.4}$$

is the necessary and sufficient condition for a permissible solution to exist.

Hence, using the given value of $y \in (0, 1/2)$ or $y_* \in (1/2, 1)$, it is possible to establish uniquely the critical flow parameters D, u_c, w_c for the corresponding range of the mean velocity $u_m (u_m > 0)$, which is defined by inequalities (3.3) or (3.4).

As was noted above, for small amplitude periodic roll waves to exist in the neighbourhood of the critical point $y < 1/2$. It is sufficient to satisfy the condition $F'(y) > 0$. Similarly, when $y_* > 1/2$, small-amplitude roll waves will exist when $F'(y_*) > 0$ and, unlike the waves in the neighbourhood of the point $y < 1/2$, these waves will satisfy the condition $dh/d\zeta < 0$ in the continuity sections (the transition A'E'B' in Fig. 1a).

4. ROLL WAVES OF LIMITING AMPLITUDE

Suppose the critical depth $y < 1/2$ is fixed and roll waves of infinitesimal amplitude exist in the neighbourhood of this critical depth, that is, $\Delta(y) = 0$, $F(y) = 0$, $\Delta'(y) > 0$, $F'(y) > 0$. As in the case of a single-layer flow, the minimum depth of the wave h^+ can be chosen as the second independent parameter, characterizing a roll wave. The conjugate depth h^- (the transition AB in Fig. 1a), which is a maximum in the period of the wave, is determined from the relations on the discontinuity. When $y < 1/2$, for a periodic solution to exist, it is necessary that

$$F(h) < 0 \text{ when } h^+ \leqslant h < y$$
$$F(h) > 0 \text{ when } y < h \leqslant h^- \qquad (4.1)$$

As can be seen from Fig. 1(a), when h^+ decreases, the depth h^- increases and inequalities (4.1) are no longer satisfied if the function $F(h)$ vanishes at one of the ends of the segment h^+, h^-. In this case, the length of the roll wave tends to infinity and, by analogy with the structure of the flow in a single-layer fluid, we are concerned with a wave of limiting amplitude.

Another situation occurs if conditions (4.1) are satisfied in the interval h^+, h^- up to the value $h^- = y_*$. Here, we are concerned with a wave of maximum amplitude $\delta h = y_* - h^l$ (Fig. 1b) in which the flow is critical immediately behind the discontinuity since $\Delta(y_*) = 0$. If $F(y_*) > 0$, then, in the neighbourhood of the level $h = y_*$, the derivatives of the solution of Eq. (2.2) increase without limit.

It has been shown in the case of flows of a single-layer fluid in an open channel [7] that roll wave packets are stable over a fairly narrow range of wave numbers and that limiting amplitude waves are not stable. The stability analysis of roll waves has still not been carried out in the case of a two-layer fluid, although the modulation equations for the variables y and h^+ can easily be obtained by the averaging Eqs (1.2) in the same way as in the case of a single-layer fluid.

The existence of roll waves with an amplitude exceeding δh is impossible. However, on reaching a maximum amplitude of the wave, it is possible to change to a fundamentally new solution of the travelling wave type which, within the framework of the two-layer flow model being considered here, will be slug flow conditions. In fact, if the condition for a maximum-amplitude wave to exist is satisfied (inequalities (4.1) hold on h_l, y_*) and, moreover

$$F(h) > 0 \text{ when } y_* < h < h_r$$

that is, the functions $\Delta(h)$ and $F(h)$ have different signs in the segment BC (Fig. 1b), it is possible to construct a new periodic solution which corresponds to the transition ABCEA. This solution is a travelling wave with a velocity D corresponding to two segments of the continuous solution (2.1) where the depth increases along AE and decreases with respect to ξ along CB (Fig. 2). Furthermore, this solution contains a pair of discontinuities of the "hydraulic jump – pit" type joining the segments of continuous flow. Since the flow is supercritical along the segments AE and CB and critical at the points B and E behind the discontinuities, the conditions for the discontinuities to be stable are satisfied and we are dealing with the "anomalous" flow conditions which have been considered previously in [10, 11]. "Anomalous" supercritical flow conditions containing a system of "hydraulic jump – pit" discontinuities were found in [10, 11] in the case of the two-layer fluid flow in the neighbourhood of a local constriction of the channel. Note that friction on the walls of the channel, which is similar to the action of a local channel constriction on a flow, plays a role in the formation of the flow structure. Here, by virtue of the smallness of the friction coefficient, this effect manifests itself in exceedingly large spatial scales.

An important feature of the "anomalous" solution ABCEA which has been constructed should be pointed out. For such a structure to exist, the condition $F(y) = 0$ as well as the condition $F(y_*) = 0$ cease to be necessary and the solution ABCEA, generally speaking, does not describe a roll wave. For a periodic solution of the above-mentioned configuration to exist when there is a further change in the flow parameters, it is therefore necessary to ensure that the following inequalities are satisfied (Fig. 1b)

$$F(h) < 0 \text{ when } h_l \leqslant h < y$$
$$F(h) > 0 \text{ when } y_* < h \leqslant h_r$$

Note that the structure of the "anomalous" solutions is determined as usual by two independent parameters, although different from those in the case of roll waves. This last remark enables us to derive modulation equations for the stability analysis of slug flow conditions using a technique which has been described previously [7, 8].

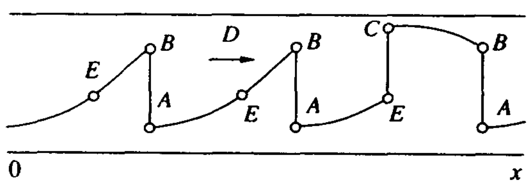

Fig. 2

Due to the complex structure of the anomalous solutions, the investigation of the hyperbolic character of the corresponding modulation equations is still an unsolved problem. Another approach, which enables one to judge the non-linear stability of roll waves and slug flow conditions, lies in an analysis of the non-stationary problem of the formation and development of finite-amplitude waves in two-layer flow as a result of the instability of uniform flow. Of course, we are referring here to a numerical investigation of this stability.

The results of a numerical calculation of the development of the instability of shear flow and the transition to slug flow conditions within the framework of model (1.1) are presented in Fig. 3. In the entrance segment of the channel ($H = 1$, $L = 50$, L is the channel length), the constant velocities of the upper and lower layers ($u_0^+ = 1.5$, $u_0^- = 2$) and the small harmonic perturbations in the depth of the lower layer ($h = 0.5 + 0.01 \cos(10t)$) are given on the left. These perturbations develop quite slowly in the shear flow of the fluid almost up to the middle of the channel. An explosive growth then occurs and there is a non-linear stage in the development of the perturbations. Since the flow being considered is supercritical over the whole channel length, the solution at fairly long times is completely determined by the Cauchy data at the channel inlet.

The position of the interface of the layers at the instant of time $t = 25$, which is sufficiently long for a quasi-steady flow pattern to become established, is shown in Fig. 3. It can be seen that, in the segment 1–1′ as a result of non-linear interaction of the waves, the channel is almost shut off, sometimes by one layer and sometimes by the other, and conditions are created for the transition to "anomalous" flow conditions. In the segment 2–2′, the quasiregular wave pattern which contains a pair of discontinuities of the "hydraulic jump – pit" type has already been completely formed, which is evidence of the stability of the slug flow conditions. The wave structure in this segment is qualitatively similar to that represented in Fig. 2 (ABCEA). A phase velocity and wavelength comparison with the corresponding stationary solutions will be possible after a two-parameter family of "anomalous" waves has been constructed, which is similar to that presented above in the case of roll waves in two-layer flows.

If, for clarity, the computed data are presented in dimensioned variables for a "water–kerosene" system ($b = 2.5$ m/s^2): $L = 1000$ m is the channel length, $H = 0.06$ m is the channel depth, $u_0^+ = 0.8$ m/s and $u_0^- = 0.6$ m/s are the constant velocities of the upper and lower layers, $h_0 = 0.03$ m is the depth of the lower layer at the channel inlet and $c_w = 0.004$, $c_i = 0$, then it can be seen that the horizontal scale is enlarged considerably compared with the structure of "anomalous" waves over a barrier in the case of the similar system in [11]. Note that the non-stationary numerical calculation was carried out using Godunov's scheme, which has previously been employed [11] to calculate "anomalous" conditions in two-layer flows over an obstacle and to compare them with experimental data.

Note that the transition from stratified flow conditions to slug flow conditions is indicative of the instability of the "intermediate" roll waves over the whole range of amplitudes up to the maximum. However, the last assertion must be considered as an hypothesis. Only the stability analysis of roll waves of finite amplitude will enable one to determine the stability and instability domains or travelling periodic waves.

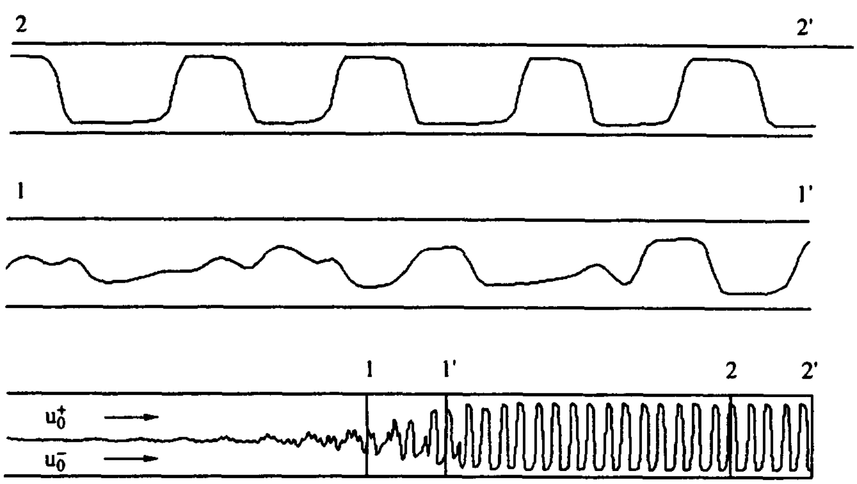

Fig. 3

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